

ERGODICITY AND ASYMPTOTIC STABILITY OF FELLER SEMIGROUPS ON POLISH METRIC SPACES

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ABSTRACT. We provide some sharp criteria for studying the ergodicity and asymptotic stability of general Feller semigroups on Polish metric spaces. As application, the 2D Navier-Stokes equations with degenerate stochastic forcing will be simply revisited.

1. INTRODUCTION

There is a vast literature of studying the ergodicity and asymptotic stability for various semigroups from dynamical systems and Markov chains. Abundant theories and applications have been established for compact or locally compact state spaces. However, it is difficult to extend them freely to infinite dimensional normed spaces or more general Polish metric spaces.

Actually, in the field of stochastic partial differential equations (SPDE for short), the uniqueness of ergodic measure can be derived from the *strong Feller* property besides *topological irreducibility*, which has been a routine to deal with the equations with non-degenerate additive noise. Recently, the *asymptotic strong Feller* property, as a celebrating breakthrough, was presented by Hairer and Mattingly [9] to solve the unique ergodicity for 2D Navier-Stokes equations with degenerate additive noise. They obtained such kind of property due to the gradient estimate (for some $\delta > 0$)

$$(1.1) \quad |\nabla P_t \varphi(x)| \leq C(x) (\|\varphi\|_\infty + e^{-\delta t} \|\nabla \varphi\|_\infty), \quad \forall t \geq t_0.$$

Similar arguments worked for semilinear SPDEs too, see Hairer and Mattingly [10]. However, sometimes the asymptotic strong Feller property is not easy to be verified (or even fails). For example, Röckner, Zhu and Zhu [14] encountered such an obstacle in the study of stochastic quasi-geostrophic equations.

There were also other notable contributions to this subject, which came from Lasota and Szarek [12] and their subsequent works for *equicontinuous* semigroups, or called *e-chains*. For many known SPDEs on Banach spaces including stochastic 2D Navier-Stokes equations, the associated transition semigroups are all equicontinuous indeed, which can be derived from the gradient estimate

$$(1.2) \quad |\nabla P_t \varphi(x)| \leq C(x) (\|\varphi\|_\infty + \|\nabla \varphi\|_\infty), \quad \forall t \geq t_0.$$

If $C(x)$ is uniformly finite in a neighborhood of x , $|P_t \varphi(x) - P_t \varphi(y)|$ can be controlled by $d(x, y)$ uniformly for all y near x and all time $t \geq t_0$, i.e. the equicontinuity.

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However, there exist non-equicontinuous semigroups, or it is too complicated to prove equicontinuity. For example, Funaki and Spohn [8] introduced the Ginzburg-Landau $\nabla\varphi$ interface model, which admits at most one shift-invariant measure. Their argument is to control the long time average of semigroup evolution on initial distributions in the L^p -Wasserstein distance (see [8, Proposition 2.1], where p was chosen to be 2), formally like

$$(1.3) \quad \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T W_p(P_t^* \mu, P_t^* \nu)^p dt \leqslant C W_p(\mu, \nu)^p.$$

To authors' knowledge, it seems hopeless to improve (1.3) uniformly in T , because the individual ergodic theorem played a fundamental role in their proof so that it could not give any uniform rate of convergence for arbitrary μ and ν . Some new developments on unique ergodicity of Ginzburg-Landau $\nabla\varphi$ interface model can be found in Cotar, Deuschel and Müller [6], Cotar and Deuschel [5].

Therefore, to reach the unique ergodicity for various models, we can attempt to skirt around the equicontinuity and asymptotic strong Feller property. What's more in the theoretical sense, these criteria stay far from necessity to prove the existence of invariant distributions. So in this paper, our purpose is to find sharp criteria for the ergodicity and asymptotic stability of *Feller semigroups* on Polish metric spaces with full generality. To this end, we will introduce some new notions, especially the *eventual continuity of semigroups* (see (1.7) and (1.8) below), which is almost necessary to the ergodic behavior. When X is a Banach space, the eventual continuity can be derived from a weak gradient estimate, formally like

$$(1.4) \quad \limsup_{t \rightarrow \infty} |\langle \nabla P_t \varphi(x), h \rangle| \leqslant C(x) (\|\varphi\|_\infty + \|\nabla \varphi\|_\infty), \quad \forall \|h\| = 1.$$

Comparing (1.4) with (1.2), one can see the new ingredient is there doesn't request a uniform t_0 for all h . In other words, the evolution allows sensitive dependence on directions. More generally, (1.3) provides a useful and essential approach to yield the eventual continuity of semigroups on Polish metric spaces.

The basic setting is as follows. Let X be a Polish space equipped with a Feller transition kernel $P(x, dy)$ on the Borel σ -field \mathcal{B} satisfying $P(x, X) \equiv 1$, and C_b the set of bounded continuous functions on X . More precisely, the Feller property means $Pf \in C_b$ for any $f \in C_b$. In many cases, it is enough to replace C_b by a separable subalgebra \mathcal{F} , namely a subalgebra in C_b separating all the points in X . For instance, all bounded Lipschitz functions form a separable subalgebra.

We call a probability measure ν an *invariant distribution* if $\nu P = \nu$, where define $\nu P(f) = \nu(Pf) := \int P f d\nu$. For simplicity, denote by O_z a neighborhood of z , $Q^m = \frac{1}{m} \sum_{n=1}^m P^n$ the m -th average kernel, and \xrightarrow{w} the weak convergence.

Let's give an overview of main topics and results in this paper.

Equicontinuity vs. eventual continuity

The equicontinuity of P^n at some $z \in X$ means for any $f \in C_b$ (or a separable subalgebra \mathcal{F})

$$\inf_{O_z} \sup_{n \geqslant 1} \sup_{y \in O_z} |P^n f(y) - P^n f(z)| = 0.$$

Note that for Feller semigroups, this is equivalent to

$$(1.5) \quad \inf_{O_z} \limsup_{n \rightarrow \infty} \sup_{y \in O_z} |P^n f(y) - P^n f(z)| = 0,$$

which implies that Q^m is equicontinuous at z too

$$(1.6) \quad \inf_{O_z} \limsup_{m \rightarrow \infty} \sup_{y \in O_z} |Q^m f(y) - Q^m f(z)| = 0.$$

Roughly speaking, the equicontinuity describes that the orbits starting from a small ball should keep close to each other uniformly in time.

We define that P^n is *eventually continuous at z* if for any $f \in C_b$ (or \mathcal{F})

$$(1.7) \quad \inf_{O_z} \sup_{y \in O_z} \limsup_{n \rightarrow \infty} |P^n f(y) - P^n f(z)| = 0.$$

which implies that Q^m is eventually continuous at z too

$$(1.8) \quad \inf_{O_z} \sup_{y \in O_z} \limsup_{m \rightarrow \infty} |Q^m f(y) - Q^m f(z)| = 0.$$

Thus $\limsup_{n \rightarrow \infty} P^n f$ and $\limsup_{m \rightarrow \infty} Q^m f$ are continuous at z , and so are their inferior limits respectively. That is why we call it the eventual continuity. In comparison with the equicontinuity, clearly hold

$$(1.5) \Rightarrow (1.7), \quad (1.6) \Rightarrow (1.8),$$

but not vice versa. The eventual continuity is much weaker since there is no uniform restriction on time.

Remark 1.1. *Here is a toy example to show that an eventual continuous semigroup can be not equicontinuous. Let \mathbb{H} be a Hilbert space with a basis $\{h_n\}_{n \geq 1}$. Let*

$$\lambda_{n,k} = \begin{cases} 2^{k-n}, & 0 \leq k < n; \\ k - n + 1, & n \leq k < 2n; \\ (k - 2n + \pi)^{-1}, & 2n \leq k < 2^n. \end{cases}$$

Denote by $x_{n,k} = \lambda_{n,k} h_n$, $X_n = \{x_{n,k} : 0 \leq k \leq 2^n\}$ the n -th state set in direction h_n , and $X = \{0\} \cup X_1 \cup X_2 \cup \dots$ the state space. Define the transition

$$P(0,0) = 1, \quad P(x_n, k \bmod 2^n, x_n, k+1 \bmod 2^n) = 1.$$

Each X_n is a cycle. Then Q_m is eventually continuous at 0, but not equicontinuous.

Remark 1.2. *Most recently, Prof. N. Bouleau told us that the notion of eventual continuity is adapted to a general framework of sticky convergence or sticking topology, which was presented in his early work [2] (or see [3]). The sticky convergence is finer than the pointwise convergence and coarser than the locally uniform convergence, which gives the coarsest topology preserving continuity.*

Let's point out, in view of the mean ergodic theorem, (1.8) is almost necessary to the existence of ergodic measures except negligible sets. Hence, it is reasonable to call (1.8) a sharp condition in the sense that there would be no information on negligible set of invariant distribution prior to we could prove its existence.

Existence of invariant distributions

First of all, let's recall a fundamental characterization of the existence of invariant distributions, for example, see [12, Proposition 3.1].

Proposition 1.3. ([12]) *Suppose there exist some point x and compact set K such that*

$$\limsup_{m \rightarrow \infty} Q^m(x, K) > 0.$$

Then there exists an invariant distribution.

There are some other “compact”-type criteria. For example, Prof. M.-F. Chen [4, Theorem 4.11] offered one for the existence of stationary distributions for Feller semigroups. Say a nonnegative measurable real-valued function h is compact if the set $\{x : h(x) \leq c\}$ is compact for all $c \geq 0$. Then there exists a stationary distribution provided there exist a compact h , some point x_0 and a constant $C \geq 0$ such that $\frac{1}{m} \sum_{n=1}^m P^n h(x_0) \leq C$ for all $m \geq 1$.

However, to study complicated models on Polish spaces, it is too difficult to determine a compact set under the infinite dimensional topology. In general, it is much more natural and useful to replace K by a neighborhood O . For equicontinuous semigroups, Szarek [15, Proposition 2.1] made some notable improvements. According to [15], say a *lower bound condition* holds at z , if for any neighborhood O_z , there exists some x such that

$$(\mathcal{L}) \quad \limsup_{m \rightarrow \infty} Q^m(x, O_z) > 0.$$

Proposition 1.4. ([15]) *Suppose P^n is equicontinuous at some z and satisfies (\mathcal{L}) at z too. Then the sequence $\{Q^m(z, \cdot)\}_{m \geq 1}$ is tight.*

Remark 1.5. *The tightness implies there is a compact K with $\limsup_{m \rightarrow \infty} Q^m(z, K) > 0$, which yields the existence of invariant distributions by Proposition 1.3.*

Condition (\mathcal{L}) is necessary to the existence of invariant distributions, but the equicontinuity (see (1.5) or (1.6)) is not. In this paper, we deal with general Feller semigroups with eventual continuity rather than equicontinuous ones. We give two sharp criteria.

Theorem 1.6. *Suppose Q^m is eventually continuous at some z , and satisfies for any O_z*

$$(\mathcal{L}_S) \quad \limsup_{m \rightarrow \infty} Q^m(z, O_z) > 0.$$

Then the sequence $\{Q^m(z, \cdot)\}_{m \geq 1}$ is tight.

Remark 1.7. (\mathcal{L}_S) is a necessary condition for the existence of invariant distributions too, even if it is a bit stronger than (\mathcal{L}) .

The next criterion has no restrictions on starting points of transitions.

Theorem 1.8. *Suppose Q^m is eventually continuous at some z and satisfies the following property:*

(\mathcal{L}_{loc}) *the lower bound condition holds at every point in a neighborhood of z .*

Then the sequence $\{Q^m(z, \cdot)\}_{m \geq 1}$ is tight.

Remark 1.9. (i) *If there exists an invariant distribution μ , we define $Y = \text{Supp} \mu$ with the relative topology. Then (\mathcal{L}_{loc}) is true for the whole Y . In other words, (\mathcal{L}_{loc}) is a necessary condition on the ergodic component. (ii) *By the Feller property, it is sufficient to assume (\mathcal{L}) at z , and assume for any $O_y \subset O_z$, there exists a time n (depending on y) such that $P^n(z, O_y) > 0$. Then (\mathcal{L}_{loc}) holds at z .**

Uniqueness of invariant distribution

It's known that there exists an ergodic measure if one can find an invariant distribution, see Hille and Worm [11, Corollary 4.8]. For this reason, we discuss the uniqueness of ergodic measure now.

According to [9], an increasing sequence of (pseudo) metrics d_i on X is called a *totally separating system* if $\lim_{n \rightarrow \infty} d_i(x, y) = 1$ for all $x \neq y$. Denote

$$\|\varphi\|_{d_i} = \sup_{x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d_i(x, y)}, \quad \|\mu - \nu\|_{d_i} = \sup_{\|\varphi\|_{d_i} \leq 1} \left| \int \varphi d\mu - \int \varphi d\nu \right|.$$

Say P^n is *asymptotic strong Feller* at z if there exists a totally separating system $\{d_i\}$ and a sequence $n_i > 0$ such that

$$(1.9) \quad \inf_{r>0} \limsup_{n \rightarrow \infty} \sup_{y \in B(z, r)} \|P^{n_i}(y, \cdot) - P^{n_i}(z, \cdot)\|_{d_i} = 0.$$

It can be applied to show that, if μ and ν are two distinct ergodic measures, then $z \notin \text{Supp}\mu \cap \text{Supp}\nu$.

Clearly, (1.9) implies P^{n_i} is eventually continuous at z with respect to the d_i -Lipschitz test functions. This fact might have nothing to do with the ergodicity if these n_i are selected irregularly. However, consider the average kernel, we have

Proposition 1.10. *Suppose a subsequence Q^{m_i} is eventually continuous at z . Then $z \notin \text{Supp}\mu \cap \text{Supp}\nu$, when μ and ν are two distinct ergodic measures.*

Consequently, the uniqueness of ergodic measures can be derived from eventual continuity on the whole X combining with the *weak type of irreducibility*, i.e. for any x_1 and x_2 , there exists some y such that for any neighborhood O_y , there exist n_1 and n_2 respectively with $P^{n_1}(x_1, O_y) > 0$ and $P^{n_2}(x_2, O_y) > 0$.

In summary, the eventual continuity (1.8) can make contributions to both the existence and uniqueness of invariant distribution.

Asymptotic stability

Now, suppose (X, \mathcal{B}, P) admits an ergodic measure μ . Simply write $X_\mu = \text{Supp}\mu$.

For equicontinuous semigroups, Szarek [16, Theorem 2] proved that

Proposition 1.11. ([16]) *Suppose P^n is equicontinuous on X , and there exists some $z \in X_\mu$ such that for any neighborhood O_z*

$$(1.10) \quad \liminf_{n \rightarrow \infty} P^n(z, O_z) > 0.$$

Then $P^n(x, \cdot) \xrightarrow{w} \mu$ for all $x \in X_\mu$.

This result can be essentially improved for general Feller semigroups. First of all, let's introduce a notion of *aperiodicity*, which is much weaker than (1.10). Say z is aperiodic, if for any O_z , there exists N such that $P^n(z, O_z) > 0$ for all $n \geq N$. We prove that

Theorem 1.12. *Suppose P^n is eventually continuous on X_μ . The next two statements are equivalent:*

- (1) $P^n(x, \cdot) \xrightarrow{w} \mu$ for all $x \in X_\mu$.
- (2) X_μ contains an aperiodic point.

Remark 1.13. *The asymptotic stability on X_μ implies a restricted eventual continuity on the subspace $(X_\mu, \mathcal{B}|_{X_\mu}, P|_{X_\mu})$ with relative topology. Hence, the eventual continuity is necessary to the asymptotic stability on $(X_\mu, \mathcal{B}|_{X_\mu}, P|_{X_\mu})$.*

It's known that $X_\mu = X$ can be derived from the *topological irreducibility*, i.e. for any $x, y \in X$ and any O_y , there exists some n with $P^n(x, O_y) > 0$. However, sometimes the topological irreducibility might be not true. In general, we provide a criterion for the global asymptotic stability.

Theorem 1.14. *Suppose that P^n is eventually continuous on X_μ and $P^n(x, \cdot) \xrightarrow{w} \mu$ for all $x \in X_\mu$. The next two statements are equivalent:*

- (1) $P^n(x, \cdot) \xrightarrow{w} \mu$ for all $x \in X$.
- (2) *there exists $z \in X_\mu$ such that for any O_z , there exists $\eta > 0$ satisfying*

$$\inf_{x \in X} \limsup_{n \rightarrow \infty} P^n(x, O_z) \geq \eta.$$

We give another criterion for the case that one can prove a local topological irreducibility around some $z \in X_\mu$ (thus z becomes an inner point in X_μ).

Theorem 1.15. *Suppose that P^n is eventually continuous on X_μ and $P^n(x, \cdot) \xrightarrow{w} \mu$ for all $x \in X_\mu$. Suppose also*

- (A1) *there exists an inner point $z \in X_\mu$;*
- (A2) *for any bounded set A and any O_z , there exists $\eta > 0$ such that for any $x \in A$, there exists k such that $P^k(x, O_z) \geq \eta$;*
- (A3) *for any $x \in X$ and $\varepsilon > 0$, there exists a bounded set B and a subsequence $n_i \rightarrow \infty$ such that $\limsup_{i \rightarrow \infty} P^{n_i}(x, B) \geq 1 - \varepsilon$.*

Then $P^n(x, \cdot) \xrightarrow{w} \mu$ for all $x \in X$.

Remark 1.16. (A2) and (A3) are necessary. The new ingredient is one doesn't need to find a uniform lower bound for $\limsup_{n \rightarrow \infty} P^n(x, O_z)$ as (2) in Theorem 1.14.

This paper is arranged as follows. In Section 2, we discuss the existence and uniqueness of invariant distribution. Sections 3 and 4 are respectively devoted to the asymptotic stability on the ergodic support and whole state space. In Section 5, we would like to simply revisit the unique ergodicity and prove the asymptotic stability of stochastic 2D Navier-Stokes equations according to our criteria, based on partial estimates from [7] and [9].

Let's point out, all the notions and results in Sections 2, 3 and 4 can be freely extended to continuous-time semigroup P_t correspondingly.

2. INVARIANT DISTRIBUTIONS

In this section, endow (X, \mathcal{B}) with a metric ρ since X is metrizable. Denote by $B(x, r) = \{y : \rho(x, y) < r\}$ the open ball of radius r centered at x , and $A^\varepsilon = \{x : \rho(x, A) < \varepsilon\}$ the ε -neighborhood of a set A .

2.1. Some lemmas.

Lemma 2.1. *Let $\{A_n^\varepsilon\}_{n \geq 1}$ be a sequence of mutually disjoint ε -neighborhoods of A_n . Then, for any compact set C , there exists $N > 0$ such that for all $n \geq N$*

$$C \cap A_n^{\varepsilon/2} = \emptyset.$$

Proof. Assume there exist $x_n \in C \cap A_n^{\varepsilon/2}$ for infinitely many n . Without loss of generality, assume that x_n tends to some $x \in C$. Then, we have $x \in A_n^\varepsilon$ for any n with $\rho(x_n, x) < \varepsilon/2$, which contradicts the mutual disjointness. \square

Lemma 2.2. *Let $\{A_n^\varepsilon\}_{n \geq 1}$ be a sequence of mutually disjoint ε -neighborhoods of compact A_n . Let $x \in X$ and $m_i \in \mathbb{N}$. Then for any $\eta > 0$, there exists $N > 0$ such that for all $n \geq N$*

$$\liminf_{i \rightarrow \infty} Q^{m_i}(x, A_n^{\varepsilon/4}) \leq \eta.$$

Proof. Define $\delta = \sup_{C \text{ compact}} \liminf_{i \rightarrow \infty} Q^{m_i}(x, C^{\varepsilon/4})$. Choose some $\gamma \geq 0$ and compact subset C to satisfy

$$0 \leq \delta - \gamma \leq \eta, \quad \liminf_{i \rightarrow \infty} Q^{m_i}(x, C^{\varepsilon/4}) \geq \gamma.$$

By Lemma 2.1, there exists some N such that $C^{\varepsilon/4} \cap A_n^{\varepsilon/4} = \emptyset$ for all $n \geq N$. Then, we have by the definition of δ that

$$\delta \geq \liminf_{i \rightarrow \infty} Q^{m_i}(x, C^{\varepsilon/4} \cup A_n^{\varepsilon/4}) \geq \gamma + \liminf_{i \rightarrow \infty} Q^{m_i}(x, A_n^{\varepsilon/4}),$$

which implies $\eta \geq \liminf_{i \rightarrow \infty} Q^{m_i}(x, A_n^{\varepsilon/4})$ for all $n \geq N$. \square

Lemma 2.3. *Let $A_n \in \mathcal{B}$ ($n \geq 1$) be a sequence of mutually disjoint sets. Let $x \in X$ and $m_i \geq 1$. Then for any $\varepsilon > 0$, there is $N > 0$ such that $\liminf_{i \rightarrow \infty} Q^{m_i}(x, A_N) \leq \varepsilon$. Hence, there exists m_{i_k} with $\limsup_{k \rightarrow \infty} Q^{m_{i_k}}(x, A_N) \leq \varepsilon$.*

Proof. Due to $1 \geq \liminf_{i \rightarrow \infty} Q^{m_i}(x, \bigcup_n A_n) \geq \sum_n \liminf_{i \rightarrow \infty} Q^{m_i}(x, A_n)$. \square

2.2. Proof of Theorem 1.8. The basic idea is partially from [15], but since we deal with the eventual continuity rather than equicontinuity, the proof becomes much more difficult here. For ease of reading, one can check firstly Step 1 and Step 4 provided (2.9) below.

Proof. Referring to Billingsley [1], it is enough to show for any $\varepsilon > 0$, there exists a compact set E such that $Q^m(z, E^{2\varepsilon}) \geq 1 - 2\varepsilon$ for all $m \geq 1$. We divide the proof into four steps.

Step 1. Given $\varepsilon > 0$. Denote $K_0 = \{z\}$ and $n_0 = 1$. By induction, we will find K_j and n_j as follows. For each $j \geq 1$, there exists some compact set E_j satisfying

$$E_j \supset \bigcup_{0 \leq l < j} K_l, \quad \text{and} \quad \min_{m \leq n_{j-1}} Q^m(z, E_j^{2\varepsilon}) \geq 1 - 2\varepsilon.$$

Let's introduce

$$(2.1) \quad \inf_{m > n_{j-1}} Q^m(z, E_j^{2\varepsilon}) =: 1 - 2\theta_j.$$

If $\theta_j = 0$, we stop the procedure; else there exists $n_j > n_{j-1}$ such that $Q^{n_j}(z, E_j^{2\varepsilon}) < 1 - \theta_j$, together with a compact set K_j satisfying

$$(2.2) \quad E_j^{2\varepsilon} \cap K_j = \emptyset, \quad \text{and} \quad Q^{n_j}(z, K_j) \geq \theta_j.$$

Either we can finish the proof just in finite steps, or collect a sequence of data $\{E_j, \theta_j, n_j, K_j\}$. Clearly, all K_j^ε are disjoint mutually. Define

$$f_j(y) = \rho(y, (K_j^{\varepsilon/4})^c) / \left(\rho(y, (K_j^{\varepsilon/4})^c) + \rho(y, K_j) \right),$$

which fulfills that $\|f_j\|_{\text{Lip}} \simeq 4/\varepsilon$ and $\mathbf{1}_{K_j} \leq f_j \leq \mathbf{1}_{K_j^{\varepsilon/4}}$.

Step 2. We want to select a subsequence from the above data such that it is so sparse that (2.8) below holds. This step will be cut into four parts.

Part 2.1. Denote $z_0 = z$, $s_0 = r$ and $B_0 = B(z_0, s_0)$. By Condition (\mathcal{L}_{loc}) , there exist some x_0 and $\{m_{i,0}\}_{i \geq 1}$ such that $\alpha_0 := \lim_{i \rightarrow \infty} Q^{m_{i,0}}(x_0, B_0) > 0$.

Using Lemma 2.2 yields a big j_0 such that

$$\begin{aligned} \varepsilon \alpha_0 / 16 &\geq \liminf_{i \rightarrow \infty} Q^{m_{i,0}}(x_0, K_{j_0}^{\varepsilon/4}) = \liminf_{i \rightarrow \infty} \int Q^{n_{j_0}}(y, K_{j_0}^{\varepsilon/4}) Q^{m_{i,0}}(x_0, dy) \\ (2.3) \quad &\geq \liminf_{i \rightarrow \infty} \int_{B_0} Q^{n_{j_0}} f_{j_0}(y) Q^{m_{i,0}}(x_0, dy). \end{aligned}$$

Due to the Feller property, define an open subset in B_0 as

$$A_0 = \{y \in B_0 : Q^{n_{j_0}} f_{j_0}(y) < \varepsilon/8\}.$$

It follows from (2.3) that

$$\varepsilon \alpha_0 / 16 \geq \liminf_{i \rightarrow \infty} \int_{B_0 - A_0} Q^{n_{j_0}} f_{j_0}(y) Q^{m_{i,0}}(x_0, dy),$$

which implies $\liminf_{i \rightarrow \infty} Q^{m_{i,0}}(x_0, B_0 - A_0) \leq \alpha_0/2$, and thus

$$(2.4) \quad \limsup_{i \rightarrow \infty} Q^{m_{i,0}}(x_0, A_0) \geq \alpha_0/2.$$

So A_0 is nonempty, which contains a ball B_1 of radius less than $r/2$ such that

$$Q^{n_{j_0}} f_{j_0}(y) \leq \varepsilon/8, \quad \forall y \in \overline{B_1}.$$

Inductively for each B_k ($k \geq 0$), Condition (\mathcal{L}_{loc}) yields some x_k and $\{m_{i,k}\}_{i \geq 1}$ with

$$\alpha_k := \lim_{i \rightarrow \infty} Q^{m_{i,k}}(x_k, B_k) > 0.$$

For the same reason, there exist j_k and a ball $B_{k+1} \subset B_k$ of radius less than $r/2^{k+1}$ satisfying

$$Q^{n_{j_k}} f_{j_k}(y) \leq \varepsilon/8, \quad \forall y \in \overline{B_{k+1}}.$$

Hence, we can find a common $y_0 \in \bigcap \overline{B_k}$ such that

$$(2.5) \quad Q^{n_{j_k}} f_{j_k}(y_0) \leq \varepsilon/8, \quad \forall k \geq 0.$$

Part 2.2. By Lemma 2.3, there is a subsequence $\{j_{k,1}\}_{k \geq 0} \subset \{j_k\}$ such that

$$\limsup_{k \rightarrow \infty} Q^{n_{j_{k,1}}} f_{j_0}(y_0) \leq \varepsilon/8.$$

For the same reason, there is $\{j_{k,2}\}_{k \geq 0} \subset \{j_{k,1}\}$ such that

$$\limsup_{k \rightarrow \infty} Q^{n_{j_{k,2}}} f_{j_{0,1}}(y_0) \leq \varepsilon/(8 \cdot 2).$$

By induction, we have $\{j_{k,l+1}\}_{k \geq 0} \subset \{j_{k,l}\}$ satisfying

$$\limsup_{k \rightarrow \infty} Q^{n_{j_{k,l+1}}} f_{j_{0,l}}(y_0) \leq \varepsilon/(8 \cdot 2^l).$$

For simplicity of notation, still use j_l instead of $j_{0,l}$. Recall (2.5), we obtain

$$(2.6) \quad Q^{n_{j_l}} f_{j_l}(y_0) \leq \varepsilon/8, \quad \limsup_{k \rightarrow \infty} Q^{n_{j_k}} f_{j_l}(y_0) \leq \varepsilon/(8 \cdot 2^l), \quad \forall l \geq 0.$$

Part 2.3. Since K_j^ε are disjoint mutually, there exists a big u such that

$$Q^{n_{j_0}} \sum_{k \geq u} f_{j_k}(y_0) \leq \varepsilon/8.$$

Combining with the first inequality in (2.6) yields

$$Q^{n_{j_0}}(f_{j_0} + \sum_{k \geq u} f_{j_k})(y_0) \leq \varepsilon/4.$$

Let $\hat{j}_0 = j_0$ and $\hat{j}_1 = j_u$. For the same reason, there exists $v > u$ such that

$$Q^{n_{\hat{j}_1}}(f_{\hat{j}_1} + \sum_{k \geq v} f_{j_k})(y_0) \leq \varepsilon/4.$$

Let $\hat{j}_2 = j_v$. By induction, we have $\{\hat{j}_l\}_{l \geq 0} \subset \{j_l\}$ such that for all $l \geq 0$

$$Q^{n_{\hat{j}_l}} \sum_{k \geq l} f_{j_k}(y_0) \leq \varepsilon/4.$$

For simplicity of notation, still use j_l instead of \hat{j}_l . Besides the second inequality in (2.6), we obtain

$$(2.7) \quad Q^{n_{j_l}} \sum_{k \geq l} f_{j_k}(y_0) \leq \varepsilon/4, \quad \limsup_{k \rightarrow \infty} Q^{n_{j_k}} f_{j_l}(y_0) \leq \varepsilon/(8 \cdot 2^l), \quad \forall l \geq 0.$$

Part 2.4. Based on the second inequality in (2.7), there exists a big u such that

$$Q^{n_{j_u}} f_{j_0}(y_0) \leq \varepsilon/4, \quad \limsup_{k \rightarrow \infty} Q^{n_{j_k}} (f_{j_0} + f_{j_u})(y_0) \leq (1 + 2^{-1}) \cdot \varepsilon/8.$$

For the same reason, there exists a big $v > u$ such that

$$Q^{n_{j_v}} (f_{j_0} + f_{j_u})(y_0) \leq \varepsilon/4, \quad \limsup_{k \rightarrow \infty} Q^{n_{j_k}} (f_{j_0} + f_{j_u} + f_{j_v})(y_0) \leq (1 + 2^{-1} + 2^{-2}) \cdot \varepsilon/8.$$

Let $\check{j}_0 = j_0, \check{j}_1 = j_u, \check{j}_2 = j_v$. By induction, we have $\{\check{j}_l\}_{l \geq 0} \subset \{j_l\}$ satisfying

$$Q^{n_{\check{j}_l}} \sum_{0 \leq k < l} f_{j_k}(y_0) \leq \varepsilon/4, \quad \forall l \geq 0.$$

Combining with the first inequality in (2.7) yields

$$Q^{n_{\check{j}_l}} \sum_{k \geq 0} f_{j_k}(y_0) \leq \varepsilon/2, \quad \forall l \geq 0.$$

For simplicity of notation, still use j_l instead of \check{j}_l , namely

$$(2.8) \quad Q^{n_{j_l}} \sum_{k \geq 0} f_{j_k}(y_0) \leq \varepsilon/2, \quad \forall l \geq 0.$$

Step 3. Write $j_{k,0} = j_k$. Let's repeat Step 2 by substituting s_0 to $s_1 = r/2$, then obtain some $\{j_{k,1}\}_{k \geq 0} \subset \{j_{k,0}\}$ and $y_1 \in \overline{B(z, s_1)}$ such that (similar to (2.8))

$$Q^{n_{j_{l,1}}} \sum_{k \geq 0} f_{j_{k,1}}(y_1) \leq \varepsilon/2.$$

By induction, we obtain the p -th subsequence $\{j_{k,p}\}_{k \geq 0} \subset \{j_{k,p-1}\}$ and some $y_p \in \overline{B(z, s_p)}$ for $s_p = r/2^p$ such that

$$Q^{n_{j_{l,p}}} \sum_{k \geq 0} f_{j_{k,p}}(y_p) \leq \varepsilon/2, \quad \forall l \geq 0.$$

Denote $\tilde{j}_p = j_{0,p}$, it follows that

$$(2.9) \quad Q^{n_{\tilde{j}_l}} \sum_{k \geq p} f_{j_k}(y_p) \leq \varepsilon/2, \quad \forall l \geq p.$$

In fact, (2.9) plays a crucial role for the proof. Note that $y_p \rightarrow z$.

Step 4. Let $j_0^* = \tilde{j}_0$. The eventual continuity yields some r_0^* such that for all $y \in B(z, r_0^*)$

$$\limsup_{m \rightarrow \infty} |Q^m f_{j_0^*}(z) - Q^m f_{j_0^*}(y)| \leq \varepsilon/8.$$

Due to (2.9), choose $y_p \in B(z, r_0^*)$, denoted by y_0^* . And for this p , choose some $j_1^* \in \{\tilde{j}_k\}$ with $j_1^* \geq \tilde{j}_p$.

By induction, if we have $j_0^*, j_1^*, \dots, j_u^*$, there is r_u^* such that for all $y \in B(z, r_u^*)$

$$(2.10) \quad \limsup_{m \rightarrow \infty} |Q^m \sum_{l \leq u} f_{j_l^*}(z) - Q^m \sum_{l \leq u} f_{j_l^*}(y)| \leq \varepsilon/8.$$

Choose $y_q \in B(z, r_u^*)$, denoted by y_u^* , and then $j_{u+1}^* \in \{\tilde{j}_k\}$ with $j_{u+1}^* \geq \tilde{j}_q$.

Consider the subsequence $\{j_k^*\}_{k \geq 0} \subset \{\tilde{j}_k\}$. Define $g = \sum f_{j_k^*} \in \text{Lip}_b$. Again, the eventual continuity yields some r^* such that for all $y \in B(z, r^*)$

$$(2.11) \quad \limsup_{m \rightarrow \infty} |Q^m g(z) - Q^m g(y)| \leq \varepsilon/8.$$

Fix some $y_u^* \in B(z, r^*)$ (for big u), denote $g_0 = \sum_{l \leq u} f_{j_l^*}$ and $g_1 = g - g_0$. We have

$$\begin{aligned} & \limsup_{k \rightarrow \infty} |Q^{n_{j_k^*}} g_1(z) - Q^{n_{j_k^*}} g_1(y_u^*)| \\ & \leq \limsup_{k \rightarrow \infty} |Q^{n_{j_k^*}} g(z) - Q^{n_{j_k^*}} g(y_u^*)| + \limsup_{k \rightarrow \infty} |Q^{n_{j_k^*}} g_0(z) - Q^{n_{j_k^*}} g_0(y_u^*)|, \end{aligned}$$

which is less than $\varepsilon/4$ by (2.10-2.11). Combining (2.9), we have

$$\limsup_{k \rightarrow \infty} Q^{n_{j_k^*}} g_1(z) \leq \limsup_{k \rightarrow \infty} Q^{n_{j_k^*}} g_1(y_u^*) + \varepsilon/4 \leq 3\varepsilon/4.$$

Recall (2.1-2.2) in Step 1, since for any $k > u$

$$\theta_{j_k^*} \leq Q^{n_{j_k^*}}(z, K_{j_k^*}) \leq Q^{n_{j_k^*}} f_{j_k^*}(z) \leq Q^{n_{j_k^*}} g_1(z),$$

it follows $\limsup_{k \rightarrow \infty} \theta_{j_k^*} \leq 3\varepsilon/4$. So there exist $\theta_{j_k^*} \leq \varepsilon$ and a compact $E_{j_k^*}$ such that

$$Q^m(z, E_{j_k^*}^{2\varepsilon}) \geq 1 - 2\varepsilon, \quad \forall m \geq 1.$$

The proof of tightness is completed. \square

2.3. Proof of Theorem 1.6.

Let's prove Theorem 1.6.

Proof. Recall Part 2.1 in the proof of Theorem 1.8, the starting point is fixed as z now. Using (2.4), we can find some n with $P^n(z, A_0) > 0$, which implies there exists a ball $B_1 \subset A_0$ of radius less than $r/2$ such that $P^n(z, B_1) > 0$.

By the Feller property, there exists $s > 0$ such that for any $\xi \in B(z, s)$

$$P^n(\xi, B_1) \geq \frac{1}{2} P^n(z, B_1) > 0.$$

From Condition (\mathcal{L}_S) at z , we derive

$$\limsup_{m \rightarrow \infty} Q^m(z, B_1) \geq \frac{1}{2} P^n(z, B_1) \cdot \limsup_{m \rightarrow \infty} Q^m(z, B(z, s)) > 0,$$

which can work for Part 2.2. Then we follow the remaining steps. \square

2.4. Proof of Proposition 1.10.

Proof. Assume $z \in \text{Supp}\mu \cap \text{Supp}\nu$. Choose $f \in \text{Lip}_b$ with $b = |\mu(f) - \nu(f)| > 0$. Then there exists $r > 0$ satisfying $\lim_{i \rightarrow \infty} |Q^{m_i} f(z) - Q^{m_i} f(y)| \leq b/4$ for all $y \in B(z, r)$. By the mean ergodic theorem, there exist y_1 and $y_2 \in B(z, r)$ such that

$$\mu(f) = \lim_{i \rightarrow \infty} Q^{m_i} f(y_1), \quad \nu(f) = \lim_{i \rightarrow \infty} Q^{m_i} f(y_2),$$

which implies $|\mu(f) - \nu(f)| \leq b/2$. This is a contradiction. \square

3. ASYMPTOTIC STABILITY ON SUPPORT

In this section, we give the proof of Theorem 1.12.

Lemma 3.1. *Suppose P^n is eventually continuous on X_μ . Then for any $f \in C_b$, there exist a sequence of compact set K_i with $\mu(K_i) \uparrow 1$, and a subsequence $P^{n_k} f$ uniformly converging to some g on each K_i . Moreover, extend g to be $\limsup_{k \rightarrow \infty} P^{n_k} f$ on whole X , then g is continuous on X_μ and $P^n g = \lim_{k \rightarrow \infty} P^{n+n_k} f$ μ -a.e. on X_μ .*

Proof. Let $F \subset X_\mu$ be a compact set. For any $\varepsilon > 0$ and $x \in F$, due to the eventual continuity, there exists some O_x such that for all $y \in O_x$

$$\limsup_{n \rightarrow \infty} |P^n f(y) - P^n f(x)| \leq \varepsilon/8.$$

Thus there exists a sequence of increasing subsets $E_{x,m}$ (containing x) tending to O_x such that for all $n \geq m$ and $y, y' \in E_{x,m}$

$$|P^n f(y) - P^n f(y')| \leq \varepsilon/2.$$

By the Feller property, $E_{x,m}$ can be chosen as a closed subset. Since F is compact, we can find a finite open covering $\{O_{x_1}, \dots, O_{x_p}\}$ of F , and then select a big m such that $\mu(F - \bigcup_{j=1}^p E_{x_j,m}) \leq \varepsilon\mu(F)/2$.

For convenience of notation, denote

$$p_0 = 1, \quad F_0 = F, \quad p_1 = p, \quad F_{j,1} = F \cap E_{x_j,m}, \quad F_1 = \bigcup_{j=1}^{p_1} F_{j,1}, \quad x_{j,1} = x_j.$$

Here $F_{j,1}$ is still compact. By induction, we have a net-like structure $\{F_{j,l}, p_l\}$ satisfying $1 \leq j \leq p_l$ and

- (1) $|P^n f(y) - P^n f(y')| \leq \varepsilon/2^l, \quad \forall \text{ big } n, \quad \forall y, y' \in F_{j,l};$
- (2) $\forall j, \exists i, \text{ s.t. } F_{j,l} \subset F_{i,l-1};$
- (3) $\mu(F_{l-1} - F_l) \leq \varepsilon\mu(F)/2^l$, where $F_l = \bigcup_{j=1}^{p_l} F_{j,l}$.

Define $F_* = \bigcap F_l$, we have $\mu(F_*) \geq (1 - \varepsilon)\mu(F)$. Then choose a subsequence n_k such that $P^{n_k} f$ converges at every $x_{j,l}$. Hence, using the Ascoli-Arzelà's arguments yields the uniform convergence on F_* for the family $\{P^{n_k} f\}$.

Now, choose arbitrarily a sequence of compact subsets \tilde{K}_i with $\mu(\tilde{K}_i) \uparrow 1$. By the above result, there exists a subsequence $P^{n_{k,i}} f$ uniformly converging on a compact subset $K_1 \subset \tilde{K}_1$ with $\mu(\tilde{K}_1 - K_1) \leq 2^{-1}$. Inductively for $i \geq 2$, we can always find the i -th subsequence $\{P^{n_{k,i}} f\} \subset \{P^{n_{k,i-1}} f\}$ uniformly converging on a compact subset $K_i \subset \tilde{K}_i$ with $\mu(\tilde{K}_i - K_i) \leq 2^{-i}$. Hence, we obtain that $P^{n_{k,i}} f$ uniformly converges on each K_i with $\mu(K_i) \uparrow 1$.

Denote by g the limit of $P^{n_k, k} f$ on $\bigcup K_i$, and extend it to be $\limsup_{k \rightarrow \infty} P^{n_k, k} f$ on X . Thus g is continuous on X_μ due to the eventual continuity.

For simplicity, rewrite n_k instead of $n_{k, k}$. The Fatou's lemma gives

$$\begin{aligned} \mu(g) &= \int \liminf_{k \rightarrow \infty} P^{n_k} f d\mu = \int P^n (\liminf_{k \rightarrow \infty} P^{n_k} f) d\mu \\ &\leq \int \liminf_{k \rightarrow \infty} P^{n+n_k} f d\mu \leq \int \limsup_{k \rightarrow \infty} P^{n+n_k} f d\mu \leq \int P^n g d\mu = \int g d\mu, \end{aligned}$$

which implies $P^n g = \limsup_{k \rightarrow \infty} P^{n+n_k} f = \liminf_{k \rightarrow \infty} P^{n+n_k} f$ μ -a.e. on X_μ . \square

Lemma 3.2. *Suppose P^n is eventually continuous on X_μ . Then for any $x \in X_\mu$ and any open set B with $\mu(B) > 0$, there exists k with $P^k(x, B) > 0$.*

Proof. Choose $f \in C_b$ with $0 \leq f \leq \mathbf{1}_B$ and $\mu(f) > 0$, then the ergodicity means $\lim_{n \rightarrow \infty} Q^n f(y) = \mu(f)$ for μ -a.e. $y \in X_\mu$. Since P^n is eventually continuous, $\lim_{n \rightarrow \infty} Q^n f$ is continuous on X_μ . Hence, there is k with $P^k(x, B) \geq P^k f(x) > 0$. \square

Now let's prove Theorem 1.12.

Proof. If P^n is asymptotically stable on X_μ , it is easy to prove Statement (2).

On the contrary, we follow the idea in [13, Proposition 18.4.3] with some adjustments. For any $f \in C_b$ with $|f| \leq 1$ and $\mu(f) = 0$, according to the monotonicity

$$\int |P^n f| d\mu = \int P^n(|f|) d\mu \geq \int |P^{m+n} f| d\mu,$$

define $v = \lim_{n \rightarrow \infty} \int |P^n f| d\mu$. By Lemma 3.1, there is a subsequence $P^{n_k} f$ converging to g on a μ -full set Y and g is continuous on X_μ . Thus we have for all $n \geq 1$

$$(3.1) \quad \int |g| d\mu = \lim_{k \rightarrow \infty} \int |P^{n_k} f| d\mu = v = \lim_{k \rightarrow \infty} \int |P^{n+n_k} f| d\mu = \int |P^n g| d\mu$$

by the dominated convergence theorem.

Claim that g preserves signs on Y . Otherwise, there exist two neighborhoods O_+ and O_- with positive μ -mass such that

$$\begin{aligned} \{x \in Y : g(x) > 0\} &\subset O_+ \subset \{x \in X : g(x) > 0\}, \\ \{x \in Y : g(x) < 0\} &\subset O_- \subset \{x \in X : g(x) < 0\}. \end{aligned}$$

Let $z \in X_\mu$ be an aperiodic point. We can find k_\pm by Lemma 3.2 such that

$$(3.2) \quad P^{k_+}(z, O_+) > 0, \quad P^{k_-}(z, O_-) > 0.$$

The Feller property yields a neighborhood U of z satisfying (3.2) for all $x \in U$. Then choose a big l , writing $l_+ = l - k_+$ and $l_- = l - k_-$, such that $P^{l_\pm}(z, U) > 0$ due to the aperiodicity and

$$P^l(z, O_\pm) \geq \int_U P^{k_\pm}(y, O_\pm) P^{l_\pm}(z, dy) > 0.$$

Again, the Feller property yields another neighborhood V of z with $P^l(x, O_\pm) > 0$ for all $x \in V$. It follows $|P^l g| < P^l |g|$ on V , then $\int |P^l g| d\mu < \int P^l |g| d\mu = \int |g| d\mu$, which contradicts (3.1). Hence, the above claim is true.

Consequently, we obtain

$$v = \int |g| d\mu = \mu(g) = \mu(f) = 0,$$

which implies by the Fatou's lemma again

$$1 = \lim_{n \rightarrow \infty} \int 1 - |P^n f| d\mu \geq \int 1 - \limsup_{n \rightarrow \infty} |P^n f| d\mu.$$

Hence, $\lim_{n \rightarrow \infty} P^n f \equiv 0$ on X_μ due to the eventual continuity. \square

4. GLOBAL ASYMPTOTIC STABILITY

In this section, we will prove Theorem 1.14 and 1.15.

Lemma 4.1. X_μ is an invariant set, i.e. $P^n(x, X_\mu) = 1$ for all $x \in X_\mu$ and $n \geq 1$.

Proof. Assume $P^n(x, X_\mu) < 1$ for some $x \in X_\mu$. Take $f \in C_b$ with $0 \leq f \leq \mathbf{1}_{X-X_\mu}$ and $P^n f(x) > 0$. The Feller property yields a neighborhood O_x with $P^n f(y) > 0$ for all $y \in O_x$. It follows $0 = \mu(f) \geq \int_{O_x} P^n f(y) d\mu(y) > 0$. \square

4.1. Proof of Theorem 1.14. The next lemma says that the process will stay in a neighborhood of the ergodic support eventually.

Lemma 4.2. Under the same conditions and (2) as in Theorem 1.14, for all $x \in X$ and $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P^n(x, X - X_\mu^\varepsilon) = 0.$$

Proof. Take $\varepsilon' \in (0, \varepsilon)$ and $f \in C_b$ such that $0 \leq f \leq 1$, $f = 1$ on $X - X_\mu^\varepsilon$ and $f = 0$ on $X_\mu^{\varepsilon'}$. Set $\gamma = \sup_{x \in X} \limsup_{n \rightarrow \infty} P^n f(x)$. Assume $\gamma > 0$.

Lemma 4.1 yields $\lim_{n \rightarrow \infty} P^n f(x) = 0$ for all $x \in X_\mu$. By the eventual continuity on X_μ , there exists $\delta > 0$ such that $\limsup_{n \rightarrow \infty} P^n f(x) \leq \frac{1}{4}\gamma$ for all $x \in X_\mu^\delta$. Then for X_μ^δ , Statement (2) gives some $\eta > 0$ such that $\limsup_{n \rightarrow \infty} P^n(y, X_\mu^\delta) \geq \eta$ for all $y \in X$.

Select $x_0 \in X$ with $\limsup_{n \rightarrow \infty} P^n f(x_0) \geq \gamma(1 - \frac{1}{2}\eta)$. The Fatou's lemma gives

$$\begin{aligned} \limsup_{n \rightarrow \infty} P^n f(x_0) &\leq \int \limsup_{n \rightarrow \infty} P^n f(y) dP^m(x_0, y) \\ &\leq \frac{1}{4}\gamma \cdot P^m(x_0, X_\mu^\delta) + \gamma \cdot P^m(x_0, X - X_\mu^\delta) = \gamma(1 - \frac{3}{4}P^m(x_0, X_\mu^\delta)). \end{aligned}$$

Then taking the inferior limit in m , we have

$$\gamma(1 - \frac{1}{2}\eta) \leq \gamma(1 - \frac{3}{4} \limsup_{m \rightarrow \infty} P^m(x_0, X_\mu^\delta)) \leq \gamma(1 - \frac{3}{4}\eta),$$

which contradicts oneself. Hence, $\gamma = 0$. \square

Now, let's prove Theorem 1.14.

Proof. If P^n is asymptotically stable on X , it is easy to prove Statement (2).

On the contrary, assume Statement (2) is true. Let $f \in C_b$ with $|f| \leq 1$ and $\mu(f) = 0$. Then $\lim_{n \rightarrow \infty} P^n f(y) = \mu(f) = 0$ for all $y \in X_\mu$. Fix arbitrary $x \in X$, due to the eventual continuity of $P^n f$ on X_μ , for any $\varepsilon > 0$, there exist a compact subset $K \subset X_\mu$ and $\delta > 0$ such that

$$P(x, X_\mu^\delta - K^\delta) \leq \varepsilon, \quad \text{and} \quad \limsup_{n \rightarrow \infty} |P^n f(y)| \leq \varepsilon, \quad \forall y \in K^\delta.$$

Hence, using Lemma 4.2 and Fatou's lemma yields

$$\limsup_{n \rightarrow \infty} |P^n f(x)| \leq \limsup_{n \rightarrow \infty} \int_{X_\mu^\delta} |P^{n-1} f(y)| P(x, dy) + \lim_{n \rightarrow \infty} P^n(x, X - X_\mu^\delta) \leq 2\varepsilon.$$

The asymptotic stability on X is proved. \square

4.2. Proof of Theorem 1.15.

Lemma 4.3. *Under the same conditions as in Theorem 1.15, for all $x \in X$*

$$\lim_{n \rightarrow \infty} P^n(x, X_\mu) = 1.$$

Proof. By (A1), let $U \subset X_\mu$ be a neighborhood containing z . Take $f \in C_b$ such that $0 \leq f \leq \mathbf{1}_U$ and $P^n(z, U) \geq P^n f(z) \geq \mu(f)/2 > 0$ for big n due to the weak convergence on X_μ . Then the eventual continuity yields another neighborhood V of z with $\limsup_{n \rightarrow \infty} |P^n f(x) - P^n f(z)| \leq \mu(f)/4$ for all $x \in V$. So this estimate gives us an increasing sequence of closed subset V_m such that $V_m \uparrow V$ and

$$P^n(x, U) \geq P^n f(x) \geq P^n f(z) - \mu(f)/4 \geq \mu(f)/4$$

for all $n \geq m$ and all $x \in V_m$.

On the other hand, (A2) says for any bounded set A and V , there exists $\eta_A > 0$ such that for any $x \in A$, there exists k with $P^k(x, V) \geq \eta_A$. Choose m (depending on x) with $P^k(x, V_m) \geq \eta_A/2$, we have due to $U \subset X_\mu$ that for all $n \geq m$

$$(4.1) \quad P^{n+k}(x, X_\mu) \geq P^{n+k}(x, U) \geq \int_{V_m} P^n(y, U) P^k(x, dy) \geq \mu(f)\eta_A/8 =: \beta_A.$$

Using Lemma 4.1, define an sequence of monotone functions

$$\varphi_n(x) := P^n(x, X_\mu) = \int_{X_\mu} P^n(y, X_\mu) P^n(x, dy) \leq P^{m+n}(x, X_\mu) = \varphi_{m+n}(x),$$

which implies $\varphi(x) := \lim_{n \rightarrow \infty} \varphi_n(x)$ exists. In particular, $\varphi = 1$ on X_μ and $\varphi \geq \beta_A$ on A by (4.1). The definition gives also $\varphi_{m+n} = P^n \varphi_m$, which implies $\varphi = P^n \varphi$ by the monotone convergence theorem.

Assume $\varphi(x) < 1$ for some $x \notin X_\mu$, we set $\varepsilon = (1 - \varphi(x))/2$. By (A3), there exists a bounded B such that $P^{n_i}(x, B) \geq 1 - \varepsilon$ for a sequence of n_i . Moreover, by the above discussion, $\varphi(y) \geq \beta_B$ for all $y \in B$. Then the invariance yields

$$\begin{aligned} \varphi(x) = P^{n_i} \varphi(x) &\geq \int_{X_\mu \cup (B - X_\mu)} \varphi(y) P^{n_i}(x, dy) \\ &\geq P^{n_i}(x, X_\mu) + \beta_B \cdot (P^{n_i}(x, B) - P^{n_i}(x, X_\mu)) \\ &\geq \varphi_{n_i}(x) + \beta_B \cdot (1 - \varepsilon - \varphi_{n_i}(x)) \\ &\xrightarrow{i \rightarrow \infty} \varphi(x) + \beta_B \cdot (1 - \varepsilon - \varphi(x)) > \varphi(x), \end{aligned}$$

which contradicts oneself. Hence, $\varphi \equiv 1$ on the whole X . \square

Now, let's prove Theorem 1.15.

Proof. Let $f \in C_b$ with $|f| \leq 1$ and $\mu(f) = 0$. Then we have for all $x \in X$

$$\lim_{n \rightarrow \infty} |P^n f(x)| \leq \limsup_{n \rightarrow \infty} \int_{X_\mu} |P^{n-1} f(y)| P(x, dy) + \lim_{n \rightarrow \infty} P^n(x, X - X_\mu) = 0$$

by using the weak convergence on X_μ and Lemma 4.3. \square

5. STOCHASTIC 2D NAVIER-STOKES EQUATIONS REVISITED

To get the unique ergodicity, it suffices to check two assumptions in Theorem 1.6. Set X to be a Banach space, $z = 0$, and w_t the associated stochastic process on X with $w_0 = z$. Hairer and Mattingly [9, Lemma A.1] gives a prior estimate ($\eta > 0, C > 0$)

$$(5.1) \quad \mathbb{E} \exp(\eta \|w_t\|^2) \leq C \exp(\eta e^{-\nu t} \|w_0\|^2), \quad \forall t > 0,$$

which implies by the Chebyshev inequality that for any ball $B(z, R)$ and all time t

$$P_t(z, B(z, R)^c) \leq e^{-\eta R^2} \mathbb{E} \exp(\eta \|w_t\|^2 \mathbf{1}_{\|w_t\| \geq R}) \leq C e^{-\eta R^2},$$

and thus $P_t(z, B(z, R)) \geq \frac{1}{2}$ when R is big.

Combining with E and Mattingly [7, Lemma 3.1] that for every $\gamma > 0$ there exists a time T_γ such that

$$(5.2) \quad \inf_{w \in B(z, R)} P^{T_\gamma}(w, B(z, \gamma)) > 0,$$

we have by the semigroup property

$$\begin{aligned} \limsup_{t \rightarrow \infty} Q^t(z, B(z, \gamma)) &= \limsup_{t \rightarrow \infty} \int P^{T_\gamma}(w, B(z, \gamma)) Q^t(z, dw) \\ &\geq \limsup_{t \rightarrow \infty} \int_{B(z, R)} P^{T_\gamma}(w, B(z, \gamma)) Q^t(z, dw) > 0. \end{aligned}$$

This gives the lower bound condition (\mathcal{L}_S) .

Moreover, the gradient estimate in [9, Proposition 4.3] reads

$$(5.3) \quad |\nabla P_t \varphi(w)| \leq C \exp(\eta \|w_t\|^2) (\|\varphi\|_\infty + e^{-\delta t} \|\nabla \varphi\|_\infty), \quad \forall t > 0,$$

which implies that P_t is equicontinuous, and thus eventually continuous. Therefore, we get the existence of ergodic measures. The uniqueness follows from the eventual continuity and weak irreducibility as explained in our introduction. Note that, the weak irreducibility still follows from (5.1) and (5.2).

To check the asymptotical stability on X , it is sufficient to show that z is aperiodic and Assumption (2) in Theorem 1.14 holds, which can both be quickly derived from (5.1) and (5.2) too.

We remark that, (5.3) is a crucial ingredient in [9], a very hard and very powerful estimate in the literature of stochastic 2D Navier-Stokes equations. However, if one is concerned only to the unique ergodicity and asymptotic stability, the contraction factor $e^{-\delta t}$ there will not be used.

And we have to admit that, it is indeed more interesting to find some new SPDEs which only hold weak gradient estimates, formally like (1.4), to exhibit fully the effectiveness of our criteria presented in this paper. But at least, we provide such a

possibility to establish the ergodic theory for more complicated stochastic models on infinite dimensional spaces.

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